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On (B_N, A_{N-1}) parabolic Kazhdan–Lusztig Polynomials

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1 Introduction

Kazhdan and Lusztig introduced Kazhdan–Lusztig polynomials $P_{x,y}$ indexed by two elements x and y of an arbitrary Coxeter group [4]. These polynomials are the coefficients of the change of basis from the standard basis of the Hecke algebra to Kazhdan–Lusztig basis. In [3], Deodhar introduced the concept of parabolic Kazhdan–Lusztig polynomials $P_{\alpha,\beta}^{\pm}$ for a Coxeter group. They are associated to the induced representation of the Hecke algebra by the one-dimensional representations of parabolic subgroups. Lascoux and Schützenberger gave an algorithm to compute $P_{\alpha,\beta}^{+}$ by using the binary tree (recall this is for Grassmannian permutations) [5]. Brenti gave a description of $P_{\alpha,\beta}^{-}$ via the concept of (shifted) "Dyck partition" through the analysis of R-polynomials and the poset structure of the Bruhat order [2]. Boe gave a binary tree algorithm to compute $P_{\alpha,\beta}^{+}$ for all Hermitian symmetric pairs [1]. In this paper, we study the Kazhdan–Lusztig polynomials in the case of unequal Hecke parameters for the Hermitian symmetric pair (B_N, A_{N-1}). Our analysis has the flavour of the concept of tangles and link patterns used in statistical mechanics and that of Temperley–Lieb algebra [7]. The plan of the paper is as follows. In Section 2, we introduce Kazhdan–Lusztig polynomials and their parabolic analogues. In Section 3, we introduce a concept of Ballot strips and new diagrammatic rules 0, I and II to stack these strips in a skew Ferrers diagram. After defining generating functions $Q_{\alpha,\beta}^{\pm}$ for stacking of strips, we provide the inversion relations for $Q_{\alpha,\beta}^{\pm}$. Section 4 is devoted to the analysis of Kazhdan–Lusztig polynomials $P_{\alpha,\beta}^{-}$. The point is that we are able to compute $P_{\alpha,\beta}^{-}$ directly through link patterns. Together with the inversion formula for $Q^{\pm}$, we show $Q^{\pm} = P^{\pm}$. In Section 5, we generalize the binary tree algorithm introduced in [1, 5]. This gives an alternative combinatorial algorithm for the computation of P^{+}. Further, the generating function Q^+ introduced in Section 3 is shown to be equal to the generating function of a generalized binary tree.

2

Let S_N, S_N^C be the finite Weyl groups associated with the Dynkin diagram of type A and C. Let $w = s_{i_1} \dots s_{i_r}$ be a reduced word in S_N^C. The length functions $l, l', l_N : S_N^C \to \mathbb{N}$ are defined by $l'(w) = \mathrm{Card}\{i_j : 1 \le i_j \le N-1\}, l_N(w) = \mathrm{Card}\{i_j : i_j = N\}$ and $l(w) := l'(w) + l_N(w) = r$. The symmetric group S_N of N letters is a subgroup of S_N^C. The restriction of l on S_N is the standard length function of S_N. We use a natural partial order in S_N^C, known as the (strong) *Bruhat* order. We write $w' \le w$ if and only if w' can be obtained as a subexpression of a reduced expression of w.

The Iwahori-Hecke algebra $\mathcal{H}$ of type B_N is an unital, associative algebra over $\mathbb{C}[t, t^{-1}, t_N, t_N^{-1}]$ satisfying

$$\begin{aligned}
&(T_i - t)(T_i + t^{-1}) = 0, \qquad 1 \le i \le N-1,\\
&(T_N - t_N)(T_N + t_N^{-1}) = 0,\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},\\
&T_{N-1} T_N T_{N-1} T_N = T_N T_{N-1} T_N T_{N-1},\\
&T_i T_j = T_j T_i, \qquad |i-j| > 1.
\end{aligned}$$

The set $\{T_w\}_{w \in S_N^C}$ is the standard monomial basis of $\mathcal{H}$.

We consider the two cases for the Hecke parameters (t, t_N):

Case A t and t_N are algebraically independent with the lexicographic order $t > t_N$,

Case B $t_N = t^m$ with some positive integer m.

We denote $t^{l'(w)} t_N^{l_N(w)}$ for Case A, and $t^{l'(w)+ml_N(w)}$ for Case B by $\mathbf{t}^{l(w)}$.

We define the bar involution of $\mathcal{H}$, $\mathcal{H} \ni a \mapsto \bar{a}$ by $T_i \mapsto T_i^{-1}, 1 \le i \le N$ together with $t^p \mapsto t^{-p}$ for $p \in \mathbb{N}_+$ (for Case A and B) and $t_N \mapsto t_N^{-1}$.

We consider the abelian group $\Gamma^A = \{t^i t_N^j | i, j \in \mathbb{Z}\}$ and $\Gamma^B = \{t^i | i \in \mathbb{Z}\}$. Introduce the lexicographic order $\Gamma^X = \Gamma_+^X \cup \{1\} \cup \Gamma_-^X$ $(X = A, B)$ where

$$\begin{aligned}
\Gamma_+^A &:= \{t^i t_N^j | i > 0, j \in \mathbb{Z}\} \cup \{t_N^i | i > 0\},\\
\Gamma_+^B &:= \{t^i | i > 0\}.
\end{aligned}$$

Theorem 1 ([6]). *There exists a unique basis $\{C_w : w \in S_N^C\}$ and a unique polynomial $P_{v,w}$ such that $\overline{C_w} = C_w$ and*

$$C_w = \sum_{v \leq w} \mathbf{t}^{l(v)-l(w)} P_{v,w} T_v,$$

where $\mathbf{t}^{l(v)-l(w)} P_{v,w} \in \mathbb{Z}(\Gamma_-^X)$.

2.1 The coset space

Let W^N be the left coset space S_N^C/S_N. The following objects are bijective to each other:

(i) A minimal (maximal) representative of the coset W^N.

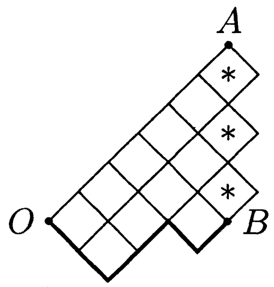
(ii) A binary string $\{1,2\}^N$. Let $\mathcal{P}_N$ be the set of binary strings in $\{1,2\}^N$.

(iii) A path from $(0,0)$ to (N,n) with $|n| \leq N$ and $N - n \in 2\mathbb{Z}$ where each step is in the direction $(1, \pm 1)$.

(iv) *A shifted Ferrers diagram* specified by a path.

We introduce the sign $\epsilon = \pm$. The maximal (resp. minimal) representatives in W^N corresponds to $\epsilon = +$ (resp. $\epsilon = -$).

Example 1. *Let $\alpha = 221121$ and $\epsilon = +$. The path α is the lowest path from O to B and the path 111111 is the up-right one from O to A. As a maximal representation in W^N, $w^+(\alpha) = s_5s_6s_2s_3s_4s_5s_6s_1s_2s_3s_4s_5s_6$. The boxes with $*$ are called anchor boxes.*



2.2 Parabolic Kazhdan–Lusztig polynomials

An element $w \in S_N^C$ is uniquely written as $w = xw'$ such that $x \in W^N$ and $w' \in S_N$. The projection $\varphi : S_N^C \to W^N$ induces two natural projections $\varphi^\pm : \mathcal{H} \cong \mathbb{C}[S_N^C] \to \mathbb{C}[W^N]$, $T_w \mapsto (\pm t^{\pm 1})^{l(w')} m_{\varphi(w)}$, where $\{m_w\}_{w \in W^N}$ is the standard basis of $\mathbb{C}[W^N]$.

Let $\alpha \in \{1,2\}^N$ be a binary string and $\mathcal{M}^\pm := \mathbb{C}[W^N]$. The action of $\mathcal{H}$ on the module $\mathcal{M}^\epsilon$ with $\epsilon \in \{+,-\}$ is given by

$$\begin{aligned} T_i m_\alpha &= \begin{cases} \epsilon t^\epsilon m_\alpha & \alpha_i = \alpha_{i+1}, \\ m_{s_i.\alpha} & \alpha_i < \alpha_{i+1}, \\ m_{s_i.\alpha} + (t - t^{-1}) m_\alpha & \alpha_{i+1} < \alpha_i, \end{cases} \quad \text{for } 1 \le i \le N-1, \\ T_N m_\alpha &= \begin{cases} m_{s_N.\alpha} & \alpha_N = 1, \\ m_{s_N.\alpha} + (t_N - t_N^{-1}) m_\alpha & \alpha_N = 2, \end{cases} \end{aligned}$$

for both Case A and B.

We introduce parabolic Kazhdan–Lusztig basis:

Theorem 2 (Deodhar). *There exists a unique basis $\{C_x^\pm\}_{x \in W^N}$ of $\mathcal{M}^\pm$ and a unique polynomial $P_{x,y}^{X,\pm}$ such that $\overline{C_x^\pm} = C_x^\pm$ and*

$$C_y^\pm = \sum_{x \le y} \mathbf{t}^{l(x)-l(y)} P_{x,y}^{X,\pm} m_x,$$

where $X \in \{A, B\}$, $P_{y,y}^\pm = 1$ and $\mathbf{t}^{l(x)-l(y)} P_{x,y}^{X,\pm} \in \mathbb{Z}(\Gamma_-^X)$.

The Kazhdan–Lusztig polynomials satisfy

Theorem 3 (Inversion formula). *Let $X \in \{A, B\}$. We have the inversion formula for $P^{X,\pm}$:*

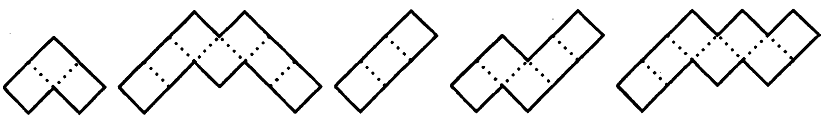
$$\sum_\alpha (-1)^{|\alpha|+|\beta|} P_{\alpha,\beta}^{X,-} P_{\alpha,\gamma}^{X,+} = \delta_{\beta,\gamma}$$

3 Combinatorics

3.1 Ballot strips

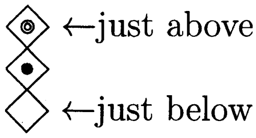
A *Ballot path* of length $(l, l') \in \mathbb{N}^2$ is a path from $(x, y) \in \mathbb{Z}^2$ to $(x + 2l + l', y + l')$ and over the horizontal line y.

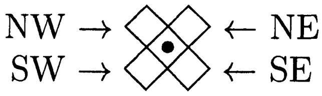
A *Ballot strip* of length $(l, l') \in \mathbb{N}^2$ is obtained by putting unit boxes (45 degree rotated) whose center are at the vertices of a Ballot path of length (l, l').



The length is $(1,0), (3,0), (0,2), (1,2)$ and $(2,2)$ from left.

We name boxes around a box as follows:





For example, the box ◈ is said to be just above the box ◈

Recall the definition of an anchor box in the skew Ferrers diagram. We put a constraint for a Ballot strip as follows.

Rule 0: Case A and B: The rightmost box of a Ballot strip of length (l, l') with $l' \geq 1$ is on an anchor box.

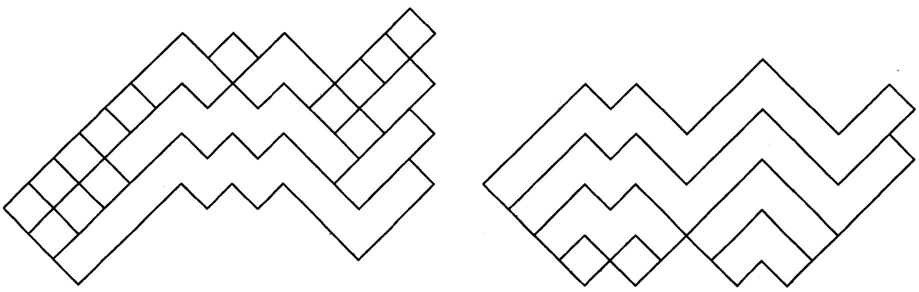
Let $\mathcal{D}, \mathcal{D}'$ be Ballot strips. We define two rules to pile $\mathcal{D}'$ on top of $\mathcal{D}$ in addition to Rule 0.

Rule I: (a) Case A & B: If there exists a box of $\mathcal{D}$ just below a box of $\mathcal{D}'$, then all boxes just below a box of $\mathcal{D}'$ belong to $\mathcal{D}$.

(b) Case B: Suppose $l' \geq m$. The number of Ballot strips of length (l, l') is even for $l' - m \in 2\mathbb{Z}$, and zero for otherwise.

Rule II: (a) Case A& B: If there exists a box of $\mathcal{D}'$ just above, NW or NE of a box of $\mathcal{D}$, then all boxes just above, NW and NE of a box of $\mathcal{D}$ belong to $\mathcal{D}$ or $\mathcal{D}'$.

(b) Case B: Suppose $l' \geq m$. If there exists a Ballot strip $\mathcal{D}$ of length (l, l') with $l' - m \in 2\mathbb{Z}$, then there is a strip of length $(l'', l' + 1), l'' \geq l$ just above $\mathcal{D}$.



Example 2.

Examples of stacks of Ballot strips satisfying Rule I (left) and Rule II (right).

Roughly speaking, Rule I (resp. Rule II) means that we are allowed to pile Ballot strips of smaller or equal (resp. longer) length on top of a Ballot strip. Further, there is at most one configuration satisfying Rule II.

3.2 Generating functions

Let $\mathcal{B}$ be a Ballot strip of length $(l,l') \in \mathbb{N}^2$. The weight $\mathrm{wt}^X(\mathcal{B})$ for a Ballot strip $\mathcal{B}$ is given by

$$\mathrm{wt}^A(\mathcal{B}) := \begin{cases} t^{2l+l'}, & l' \text{ is even,} \\ -\sigma t^{2l}t_N^2 & l' \text{ is odd.} \end{cases} \quad \textit{for Case A.}$$

$$\mathrm{wt}^B(\mathcal{B}) := \begin{cases} \sigma^{l'}t^{2l+l'}, & 0 \le l' \le m-1 \\ t^{m+2l+l'} & l' \ge m, l'-m \in 2\mathbb{Z}, \\ t^{m+2l+l'-1} & l' \ge m, l'-m-1 \in 2\mathbb{Z}, \end{cases} \quad \textit{for Case B.}$$

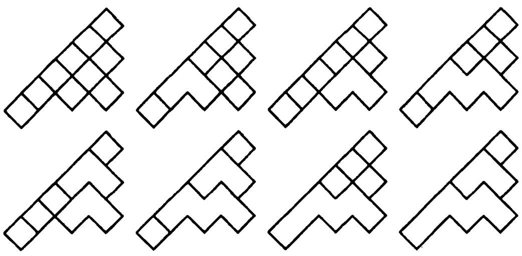
where $\sigma = +$ (resp. $-$) in case of Rule I (resp. Rule II).

Definition 1. *The generating function of Ballot strips for the paths $\alpha < \beta$ with the sign ϵ is defined by*

$$Q_{\alpha,\beta}^{X,Y,\epsilon} = \sum_{C \in \mathrm{Conf}^Y(\alpha,\beta)} \prod_{\mathcal{B} \in C} \mathrm{wt}^X(\mathcal{B}).$$

where $X \in \{A,B\}, Y \in \{I, II\}$ and $\epsilon \in \{+,-\}$. Define $Q_{\alpha,\alpha}^{X,Y,\epsilon} = 1$.

Example 3. *Let $(\alpha,\beta) = (111111, 211212)$. The possible configurations of Ballot strips for Case A and Case B ($m \ge 2$) are*



The generating functions are

$$\begin{aligned} Q_{\alpha,\beta}^{A,I,+} &= 1 + 2t^2 + 2t^4 + t^6 - s^2t^4 - s^2t^6, \\ Q_{\alpha,\beta}^{B,I,+} &= (1+t^2)^2(1+t^4), \quad m \ge 2, \\ Q_{\alpha,\beta}^{B,I,+} &= 1 + 2t^2 + 2t^4 + t^6, \quad m = 1. \end{aligned}$$

Theorem 4 (Inversion Formula). *The generating functions $Q_{\alpha,\beta}^{X,Y,\epsilon}$ satisfy*

$$\sum_\beta Q_{\alpha,\beta}^{X,I,-} Q_{\beta,\gamma}^{X,II,-} (-1)^{|\beta|+|\gamma|} = \delta_{\alpha,\gamma}$$

The outline of the proof. Let us fix a configuration of Ballot strips in the region delimited by paths α and γ. This region is divided into two by a path β. The region delimited by paths α (resp. γ) and β satisfies Rule I (resp. Rule II). Note that β depends on the configuration and there may be several possible choices of β. β is specified by choices of "boundary" strips, which can belong to the region governed either by Rule I or Rule II. We have

$$\sum_{\beta} Q^{X,I,-}_{\alpha,\beta} Q^{X,II,-}_{\beta,\gamma}(-1)^{|\beta|+|\gamma|} = \sum_{\mathcal{C}} |\mathrm{wt}(\mathcal{C})| \sum_{\beta\in\mathcal{P}(\mathcal{C})} \mathrm{sign}(\mathcal{C})(-1)^{|\beta|+|\gamma|},$$

where $\mathcal{P}(\mathcal{C})$ is the set of paths β between α and γ such that the region below β satisfy Rule I and the one above β satisfy Rule II. By taking the sum over all possible β's for the fixed configuration, we have $\sum_{\beta\in\mathcal{P}(\mathcal{C})} \mathrm{sign}(\mathcal{C})(-1)^{|\beta|+|\gamma|} = 0$. Here, We take care about the sign $\sigma = \pm$. $\square$

4 Kazhdan–Lusztig polynomials $P^{\pm}_{\alpha,\beta}$

The relations among the Kazhdan–Lusztig polynomials $P^{\pm}_{\alpha,\beta}$ and the generating functions $Q^{X,\epsilon}_{\alpha,\beta}$ that we shall establish in subsequent sections are summarized as:

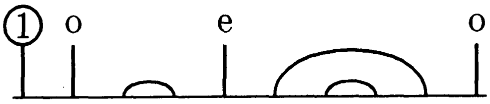
$$\begin{array}{ccc}
P^{-}_{\alpha,\beta} = Q^{II,-}_{\alpha,\beta} & \xleftrightarrow{\text{transpose}} & Q^{II,+}_{\alpha,\beta} \\
\updownarrow \text{inverse} & & \updownarrow \text{inverse} \\
Q^{I,-}_{\alpha,\beta} & \xleftrightarrow{\text{transpose}} & P^{+}_{\alpha,\beta} = Q^{I,+}_{\alpha,\beta}
\end{array}$$

4.1 Module $\mathcal{M}^{-}$: link pattern for Case A

Let $\alpha \in \mathcal{P}_N$ be a binary string of length N. We make a pair between adjacent 2 and 1 (in this order) in the string α and remove it from α. We continue this procedure until it becomes a sequence $1\dots12\dots2$. We call these remaining 1's (resp. 2's) as unpaired 1's (resp. 2's). The $(2i-1)$-th (resp. $2i$-th) unpaired 2 from the right is called as an o-unpaired (resp. e-unpaired) 2.

We introduce a graphical notation for these pairs, an unpaired 1, an e- and o-unpaired 2. Consider a line with N points. If α_i and α_j make a pair, then we connect i and j via an arch. If α_i is an unpaired 1, we put a vertical line with a circled 1. If α_i is an e-unpaired (resp. o-unpaired) 2, we put a vertical line with a mark e (resp. o). We call this graphical notation as a *link pattern* for Case A.

Example 4. *Let $\alpha = 1221222112$. The link pattern is*



Recall that the module $\mathcal{M}^-$ is spanned by the set of basis $\{m_\alpha\}_{\alpha\in\mathcal{P}_N}$. The space is isomorphic to V^N where $V \cong \mathbb{C}^2$ has the standard basis $\{|1\rangle, |2\rangle\}$. When i-th component of the tensor product is $x \in \{1,2\}$, we denote it by $|x\rangle_i$. We simply write $|xx'\rangle_{ij}$ for the tensor product $|x\rangle_i \otimes |x'\rangle_j$ and sometimes denoted by $|xx'\rangle$ if the components are obvious. Hereafter, we identify a base $m_\alpha, \alpha \in \{1,2\}^N$ with $|\alpha_1 \ldots \alpha_N\rangle$.

An arch, vertical line with e,o and a circled 1 are building blocks of a link pattern corresponding to a string $\alpha \in \{1,2\}^N$. We introduce a map ϖ^A from these building blocks to a vector in V^2 or V:

 $\mapsto \; |21\rangle + t^{-1}|12\rangle,$

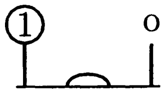
 $\mapsto \; |2\rangle + t_N^{-1}|1\rangle,$

 $\mapsto \; |2\rangle + t^{-1}t_N|1\rangle,$

 $\mapsto \; |1\rangle$

Then, we extend the map ϖ^A to a link pattern for a string α.

Example 5.

$\varpi^A(1212) =$ 

$$
\begin{aligned}
&= |1\rangle_1 \otimes (|21\rangle_{23} + t^{-1}|12\rangle_{23}) \otimes (|2\rangle_4 + t_4^{-1}|1\rangle_4) \\
&= m_{1212} + t^{-1}m_{1122} + t_4^{-1}m_{1211} + t^{-1}t_4^{-1}m_{1121}
\end{aligned}
$$

Theorem 5. *An element $\varpi^A(\alpha)$ is Kazhdan–Lusztig basis $C_\alpha^{A,-}$.*

Corollary 1.

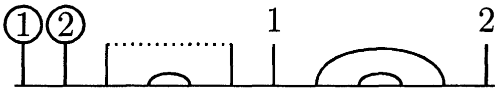
$$Q_{\alpha,\beta}^{A,II,-} = P_{\alpha,\beta}^{A,-}$$

4.2 Module $\mathcal{M}^-$: link pattern for Case B

Let $\alpha \in \mathcal{P}_N$ be a binary string. We make pairs between 2's and 1's. Then, we have remaining unpaired 1's and 2's as Case A. If α_i is the j-th $(1 \leq j \leq m)$

unpaired 2 from the right, put a vertical line with the integer $m+1-j$. If α_i and $\alpha_{i'}$ with $i < i'$ are the j-th and $(j+1)$-th unpaired 2's with $j \geq m+1$ and $j-m+1 \in 2\mathbb{Z}$, put vertical lines (on the i-th and i'-th point) whose endpoints are connected by a dotted line. If α_i is an unpaired 1 or a remaining unpaired 2 not classified above, then we put a vertical line with a circled 1 or a circled 2 respectively on the i-th point. We call this graph as a *link pattern* for Case B.

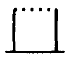
Example 6. *Let* $\alpha = 122212222112$ *and* $m = 2$. *The link pattern is*



We define the map ϖ^B from the building blocks to a vector in V or V^2:

 $\mapsto |21\rangle + t^{-1}|12\rangle,$

$\mapsto |2\rangle + (-1)^{m-p}t^{-p}|1\rangle,$

 $\mapsto |22\rangle + t^{-1}|11\rangle,$

$\mapsto |x\rangle, \quad x \in \{1,2\}.$

Together with the map from a binary string to a link pattern, we naturally extend the map ϖ^B from a binary string to a vector in $\mathcal{M}^-$, and denote it by ϖ^B.

Theorem 6. *An element* $\varpi^B(\alpha)$ *is Kazhdan–Lusztig basis* C^-_α.

Corollary 2.

$$Q^{B,II,-}_{\alpha,\beta} = P^-_{\alpha,\beta}.$$

4.3 Module $\mathcal{M}^+$: Case A & B

We prove that the generating functions $Q^{X,II,-}_{\alpha,\beta}$, $X = A, B$ are equal to the Kazhdan–Lusztig polynomials $P^-_{\alpha,\beta}$. The generating function $Q^\pm_{\alpha,\beta}$ satisfy the inversion relation which is exactly the same as the inversion formula (Theorem 3). Therefore, we have

Theorem 7.

$$Q^{X,I,+}_{\alpha,\beta} = P^+_{\alpha,\beta}.$$

5 Binary tree

Let $\mathcal{Z}$ be a set such that $\emptyset \in \mathcal{Z}$, $z \in \mathcal{Z} \Rightarrow 1z2 \in \mathcal{Z}$ and if $z_1, z_2 \in \mathcal{Z}$ then the concatenation $z_1z_2 \in \mathcal{Z}$.

A binary string α is of the form $\underline{2}z_1\underline{2}z_2 \ldots \underline{2}z_p\underline{1}z_{p+1}\underline{1} \ldots \underline{1}z_q$ for some integer $p, q \geq 0$ with $z_i \in \mathcal{Z}$. We call an underlined 1 (resp. 2) as an unpaired 1 (resp. 2).

We denote by $||\alpha||$ the length of a binary string α and by $||\alpha||_\sigma$ the number of σ in the string α. Let $\alpha = \alpha'vw\alpha''$ and $\beta = \beta'\underline{12}\beta''$ with $||\alpha'|| = ||\beta'||$, $v, w \in \{1, 2\}$. A *capacity* of the edge corresponding to the underlined 1 and 2 in β is defined by

$$\mathrm{cap}(12) := ||\alpha'v||_1 - ||\beta'1||_1.$$

Let $\alpha = \alpha'v$ and $\beta = \beta'\underline{1}$. Similarly, the capacity of underlined 1 is defined by

$$\mathrm{cap}(1) := ||\alpha||_1 - ||\beta||_1.$$

Note that the condition $\alpha \leq \beta$ implies a capacity is always non-negative.

The capacity of β with respect to α is the collection of capacities of pairs of adjacent 1 and 2 in α and that of the rightmost 1 in β if it exists.

5.1 Case A

We divide unpaired 1's into two classes. In α, the $(2i-1)$-th (resp. $2i$-th) unpaired 1 from the right is called o-unpaired (resp. e-unpaired) 1.

A binary tree $A(\alpha)$ satisfies

(◇1) $A(\emptyset)$ is the empty tree.

(◇2) $A(2w) = A(w)$.

(◇3) $A(zw)$, $z \in \mathcal{Z}$ is obtained by attaching the tree for $A(z)$ and $A(w)$ at their roots.

(◇4) $A(1z2)$, $z \in \mathcal{Z}$ is obtained by attaching an edge just above the tree $A(z)$.

(◇5) If unpaired 1 in $\underline{1}w$ is e-unpaired (resp. o-unpaired) 1, $A(1w)$ is obtained by attaching an edge just above the tree $A(w)$ and mark the edge with "e" (resp. "o").

The capacity of β with respect to α is written as integers on leaves of $A(\beta)$. Denote by $A(\beta/\alpha)$ a tree equipped with capacities.

A *labelling* of $A(\beta/\alpha)$ is a set of non-negative integers on edges of $A(\beta)$ satisfying

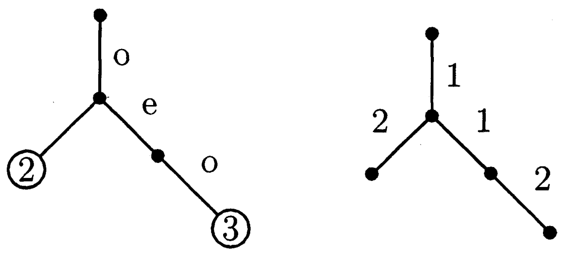
(♣1) An integer on an edge connecting to a leaf is less than or equal to its capacity.

(♣2) Integers on edges are non-increasing from leaves to the root.

Let $\sigma, \sigma_e, \sigma_o$ be the sum of labels on edges without "e" and "o", with "e", with "o".

Definition 2. *The generating function $R^A_{\alpha,\beta}$ of labellings on $A(\beta/\alpha)$ is defined by $R^A_{\alpha,\beta} = \sum_\nu t^{2\sigma}(-t_N^2)^{\sigma_o}(-t^2/t_N^2)^{\sigma_e}$, where the sum runs over all labellings of $A(\beta/\alpha)$.*

Example 7. *Let $(\alpha, \beta) = (1111111, 2211211)$. The binary tree $A(\beta)$ and a labelling is*



The capacities of a pair 12 *and o-unpaired* 2 *are* 2 *and* 3 *respectively. The weight of the labelling is $t^4 t_N^4$.*

Theorem 8.

$$Q^{A,I,-}_{\alpha,\beta} = R^A_{\alpha,\beta}$$

5.2 Case B

If α_i is the $(m+1-j)$-th $(1 \le j \le m)$ unpaired 1 from the right, we call this as j-*terminal* 1. If α_i and $\alpha_{i'}$ with $i < i'$ are the j-th and $(j+1)$-th unpaired 1's with $j \ge m+1$ and $j-m$ odd, we make a pair these 1's and call it a 11-*pair.* If α_i is an unpaired 1 and not classified above, we call this as an *extra-unpair* 1.

$A(\beta)$ is defined recursively by the following rules. The rules (◇1)-(◇4) are the same as Case A. We replace (◇5) by the following four conditions:

($\diamondsuit 5'$) If underlined 1 in $\underline{1}w$ is the j-terminal with $1 \le j \le m$, $A(\underline{1}w)$ is obtained by putting an edge just above the tree $A(w)$. Then mark this edge with a plus "+" only when $j = 1$.

($\diamondsuit 6$) Suppose underlined 1 in $\underline{1}z\underline{1}w$ is a 11-pair. The tree $A(1z1w)$ is obtained by attaching an edge above the root of $A(zw)$. We mark the edge with a plus "+".

($\diamondsuit 7$) If the underlined 1 in $\underline{1}w$ is an extra-unpair 1, we have $A(1w) = A(w)$.

($\diamondsuit 8$) When an edge e immediately "precedes" an edge e' in the binary tree $A(w)$, we put a dotted arrow from the edge e to the edge e'.

Further, we need an additional information on the tree. Suppose $w = w'z_{m+2r}1 \dots z_1 1 z_0$ with $z_i \in \mathcal{Z}$ and $r \ge 0$ (z_{m+2r} is non-empty and maximal). Set $w'' = 1z_{m+2r-1}1 \dots z_1 1 z_0$ such that $w = w'z_{m+2r}w''$ and $z_{m+2r} = x_s x_{s-1} \dots x_1$ with $x_i \in \mathcal{Z}$. Here, all x_i's can not be decomposed further into a product of non-empty elements in $\mathcal{Z}$. Then the tree $A(x_i)$ contains a unique maximal edge (the edge connecting to the root) corresponding to a pair 12. $A(w'')$ contains a unique maximal edge corresponding to a 11-pair or a 1-terminal. Observe that $A(x_i) \subseteq A(w)$, $A(w'') \subseteq A(w)$ as binary trees. We say that the maximal edge of $A(x_i)$ (resp. $A(w'')$) *immediately precedes* the maximal edge of $A(x_{i+1})$ (resp. $A(x_1)$) for $1 \le i \le s$.

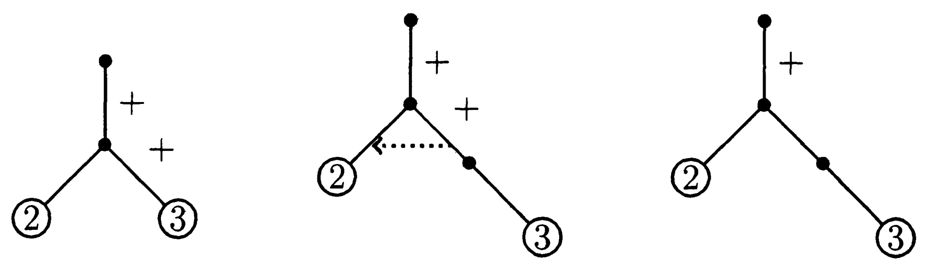
($\diamondsuit 8$) When an edge e immediately precedes an edge e' in the binary tree $A(w)$, we put a dotted arrow from the edge e to the edge e'.

In addition to ($\clubsuit 1$) and ($\clubsuit 2$) (the same as Case A), we require

($\clubsuit 3$) An integer attached to any edge with a plus "+" must be even.

($\clubsuit 4$) If the label on an edge is less than or equal to the labels on all "preceding" edges, then the former must be even.

Example 8. *Let $\alpha = 22111211$. The binary trees for α with $m = 1, 2$ and 3 from left to right.*



Given a labelling ν, let $|\nu|$ be the sum of the labels on all edges $A(\beta/\alpha)$.

Definition 3. *The generating function $R^B_{\alpha,\beta}$ of labellings on $A(\beta,\alpha)$ is defined by $R^B_{\alpha,\beta} = \sum_\nu t^{2|\nu|}$.*

Theorem 9.

$$P^{B,+}_{\alpha,\beta} = Q^{B,I,+}_{\alpha,\beta} = R^B_{\alpha,\beta}.$$

5.3 Outline of the proof of Theorems 8 and 9

Theorem 10. *There exists a bijection between labellings of $A(\beta/\alpha)$ and configurations of Ballot strips between paths α and β satisfying Rule I.*

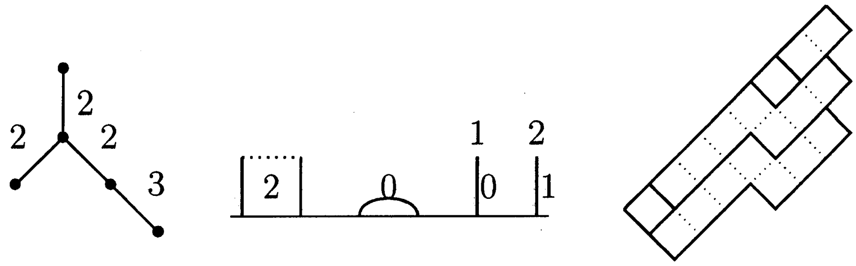


Figure 1: A bijection among a binary tree, a labelled link pattern and a configuration of Ballot strips.

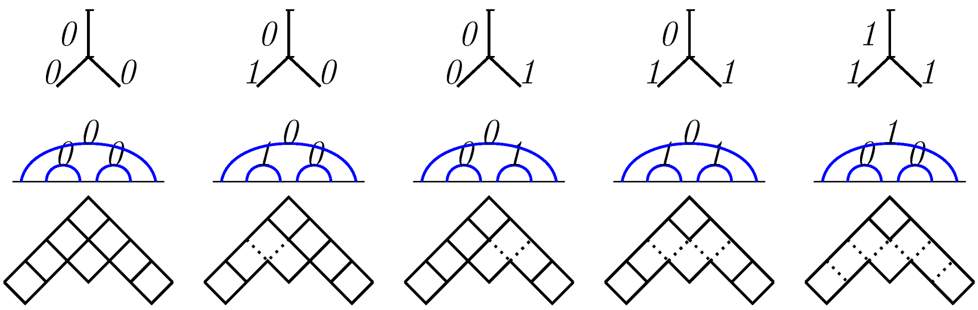
We take a "dual" graph of a binary tree $A(\beta)$ to obtain a link pattern. In Case A, an edge without a mark (resp. with "o" or "e") in a binary tree corresponds to an arch (resp. a vertical line with "o" or "e") in the link pattern. In Case B, an edge without "+" in a binary tree corresponds to an arch (corresponding to a pair 12) or a vertical line with the integer p with $2 \leq p \leq m$ in the link pattern. An edge with "+" in a binary tree corresponds to a vertical line with the integer 1 or to an arch for a paired 1's in the link pattern. Notice that the map from link patterns to trees is not one-to-one without fixing the string β: for some cases in Case B, we cannot distinguish an arch from a vertical line in a link pattern by looking at only the binary tree (see Figure 1).

An edge of the binary tree corresponds to an arch of the link pattern. We put a non-negative integer on an arch of the obtained link pattern in the following way: 1) For a given arch, we put the difference of integers on the corresponding and parent edges of $A(\beta)$. 2) On the smallest arch, the integer is less than or equal to the capacity of the corresponding leaf of $A(\beta)$. We call the link pattern with non-negative integers on arches as labelled link pattern.

Note that we have a bijection between a labelling of $A(\beta/\alpha)$ and a labelled link pattern (for a given binary string β).

We stack Ballot strips according to the labelling of the link pattern. We put a corresponding Ballot strip starting from outer arches to inner ones. Then, we merge the overlapped boxes.

Example 9. *A bijection for* $(\alpha,\beta) = (11112222, 21121221)$.



References

[1] B. D. Boe, *Kazhdan–Lusztig polynomials for Hermitian symmetric spaces*, Trans. Amer. Math. Soc. **309** (1988), 279–294.

[2] F. Brenti, *Kazhdan–Lusztig and R-polynomials, Young's lattice, and Dyck partitions*, Pacific Journal of Mathematics **207** (2002), 257–286.

[3] V. Deodhar, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials*, J. Algebra 111 (1987), no. 2, 483–506.

[4] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.

[5] A. Lascoux and M.-P. Schützenberger, *Polynômes de Kazhdan & Lusztig pour les grassmanniennes*, Young tableaux and Schur functions in algebra and geometry (Toruń 1980), Astérisque, vol. 87, Soc. Math. France, Paris, 1981, pp. 249–266.

[6] G. Lusztig, *Hecke Algebra with Unequal Parameters*, CRM monograph series, vol. 18, American Mathematical Society, 2003.

[7] H. Temperley and E. Lieb, *Relations between the "percolation" and "colouring" problem and other graph-theoretical problems with regular lattices: some exact results for the "percolation" problem*, Proc. Roy. Soc. London Ser. A **322** (1971), no. 1549, 251–280.